Solution to Problem Set 8
Optical Waveguides and Fibers (OWF)

Problem 1: Linearly polarized modes (LP) in a step-index fiber.

The modes of a step-index fiber can be calculated analytically in an *exact* form, leading to a classification in $\mathrm{TE}_{0,\mu}$, $\mathrm{TM}_{0,\mu}$ and hybrid modes ($EH_{\nu,\mu}$ and $HE_{\nu,\mu}$). When looking for exact solutions, one can find a differential equation for the $\underline{\mathcal{E}}_z$ and $\underline{\mathcal{H}}_z$ components, from which the transverse components can be derived. A simplified approximation can be used under the assumption that the mode is weakly guided ($n_1 \to n_2$) and has a dominant linearly polarized transverse field component, which - without loss of generality - we denote as $\underline{\mathcal{E}}_x$ while assuming $\underline{\mathcal{E}}_y = 0$.

Because of the assumption of weak guidance, the scalar Helmholtz equation can be used:

$$\nabla^2 \underline{\Psi}(r,\varphi) + \left(k_0^2 n^2 - \beta^2\right) \underline{\Psi}(r,\varphi) = 0, \tag{1}$$

where $\underline{\Psi}(r,\varphi)$ denotes the $\underline{\mathcal{E}}_x$ component of the mode.

a) Write Eq. (1) in cylindrical coordinates.

Solution:

By expressing the differential operator in cylindrical coordinates, Eq. (1) can be written as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \underline{\Psi}(r,\varphi)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \underline{\Psi}(r,\varphi)}{\partial \varphi^2} + \left(k_0^2 n^2 - \beta^2\right)\underline{\Psi}(r,\varphi) = 0$$

b) Separate the variables, i.e., assume that the solution can be written in the form $\underline{\Psi}(r,\varphi) = g(r)h(\varphi)$. Insert this ansatz into the result from part a), separate it into a sum of two expressions where one depends exclusively on r and the other exclusively on φ. Show that $\sin(\nu\varphi)$ and $\cos(\nu\varphi)$ are solutions for the φ-dependent part. Why must ν be an integer?

Solution:

Inserting $\underline{\Psi}(r,\varphi) = g(r)h(\varphi)$ into the equation leads to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\left(g(r)h(\varphi)\right)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\left(g(r)h(\varphi)\right)}{\partial \varphi^2} + \left(k_0^2 n^2 - \beta^2\right)\left(g(r)h(\varphi)\right) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g(r)}{\partial r}\right)h(\varphi) + \frac{1}{r^2}\frac{\partial^2 h(\varphi)}{\partial \varphi^2}g(r) + \left(k_0^2 n^2 - \beta^2\right)\left(g(r)h(\varphi)\right) = 0$$

Multiplying the whole equation with r^2 and dividing by $g(r)h(\varphi)$ we obtain:

$$\frac{r}{g(r)}\frac{\partial}{\partial r}\left(r\frac{\partial g(r)}{\partial r}\right) + \frac{1}{h(\varphi)}\frac{\partial^2 h(\varphi)}{\partial \varphi^2} + \left(k_0^2 n^2 - \beta^2\right)r^2 = 0 \tag{2}$$

If the sum is always equal to zero, then both r and φ dependent parts must be constant. We can write for the φ dependent part:

$$\frac{1}{h(\varphi)}\frac{\partial^2 h(\varphi)}{\partial \varphi^2} = C_1$$

The solutions of the last equation are $\sin(\nu\varphi)$ and $\cos(\nu\varphi)$, where $\nu^2 = -C_1$. If we consider that $h(\varphi)$ describes the azimuthal field, we can argue that for a guided mode the field must be exactly the same after one roundtrip, and therefore periodic with $\varphi = 2\pi$. Thus, ν must be an integer.

c) Insert the sinusoidal solution for $h(\varphi)$ into the result of part a) and show that the differential equation for $g(r)$ can be written as:

$$r^2\frac{\partial^2 g(r)}{\partial r^2} + r\frac{\partial g(r)}{\partial r} + \left[\left(k_0^2 n_i^2 - \beta^2\right)r^2 - \nu^2\right]g(r) = 0, \tag{3}$$

where n_1 is the core index and n_2 is the cladding index.

Solution:

Inserting $h(\varphi) = \sin(\nu\varphi)$ we obtain for Eq. (2):

$$\begin{aligned}
\frac{r}{g(r)}\frac{\partial}{\partial r}\left(r\frac{\partial g(r)}{\partial r}\right) + \frac{-\nu^2}{\sin(\nu\varphi)}\sin(\nu\varphi) + \left(k_0^2 n^2 - \beta^2\right) r^2 &= 0 \\
r\frac{\partial}{\partial r}\left(r\frac{\partial g(r)}{\partial r}\right) + \left[\left(k_0^2 n^2 - \beta^2\right) r^2 - \nu^2\right] g(r) &= 0 \\
r^2\frac{\partial^2 g(r)}{\partial r^2} + r\frac{\partial g(r)}{\partial r} + \left[\left(k_0^2 n^2 - \beta^2\right) r^2 - \nu^2\right] g(r) &= 0
\end{aligned}$$

The same solution is obtained when using $h(\varphi) = \cos(\nu\varphi)$.

Using the fact that Eq. (3) is solved by Bessel functions and modified Bessel functions, the total solution of Eq. (1) can be written as:

$$\underline{\Psi}(r,\varphi) = \begin{cases} A\mathrm{J}_\nu\left(u\frac{r}{a}\right)\cos(\nu\varphi+\psi) & \text{for } 0 \le x \le a \\ A\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\mathrm{K}_\nu\left(w\frac{r}{a}\right)\cos(\nu\varphi+\psi) & \text{for } a < x \end{cases} \tag{4}$$

where J_ν is the Bessel function of the first kind of order ν, K_ν is the decaying modified Bessel function of order $\nu = 0, 1, 2, ...$, $\psi \in \{0, \frac{\pi}{2}\}$, $u = a\sqrt{k_0^2 n_1^2 - \beta^2}$, $w = a\sqrt{\beta^2 - k_0^2 n_2^2}$.
In this relation we assumed that $\underline{\Psi}(r,\varphi)$ is continuous at $r = a$.

d) Why is this assumption legitimate?

Solution:

At this point, the approximation of a low index contrast is used. The field component $\underline{\Psi}(r,\varphi)$ can be decomposed into a φ-dependent part that is tangential to the interface at $r = a$, and a radial component that is normal to the interface at $r = a$. The tangential E-field component is always continuous at a boundary, while for the normal field component the D-field is continuous and the E-field jumps for different ε_r at the interface. With the low index contrast, we make the assumtion that the refractive index contrast is so small that the jump of the E-field is negligible, and we can assume that $\underline{\Psi}(r,\varphi)$ is continuous at $r = a$.

Starting from the equation

$$\nabla \cdot \underline{\mathbf{D}} = 0 \tag{5}$$

it is possible to show that in the limit $n_1 \to n_2$ the derivative $\frac{\partial \Psi}{\partial r}$ must be continuous as well.

e) Use this fact to derive the characteristic equation for LP-modes:

$$\frac{u\mathrm{J}'_\nu(u)}{\mathrm{J}_\nu(u)} = \frac{w\mathrm{K}'_\nu(w)}{\mathrm{K}_\nu(w)} \tag{6}$$

Solution:

If the derivative is continuous at $r = a$, we can write from Eq. 4:

$$\begin{aligned}
\frac{\partial}{\partial r}\left[A\mathrm{J}_\nu\left(u\frac{r}{a}\right)\cos(\nu\varphi+\psi)\right] &= \frac{\partial}{\partial r}\left[A\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\mathrm{K}_\nu\left(w\frac{r}{a}\right)\cos(\nu\varphi+\psi)\right] \\
\frac{\partial}{\partial r}\left[\mathrm{J}_\nu\left(u\frac{r}{a}\right)\right] &= \frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\frac{\partial}{\partial r}\left[\mathrm{K}_\nu\left(w\frac{r}{a}\right)\right] \\
\frac{u}{a}\mathrm{J}'_\nu\left(u\frac{r}{a}\right) &= \frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)}\frac{w}{a}\mathrm{K}'_\nu\left(w\frac{r}{a}\right) \\
\xrightarrow{r=a} \quad \frac{u\mathrm{J}'_\nu(u)}{\mathrm{J}_\nu(u)} &= \frac{w\mathrm{K}'_\nu(w)}{\mathrm{K}_\nu(w)}
\end{aligned}$$

f) We want now to simplify Eq. (6) by getting rid of the derivative of the Bessel function. For this purpose, make use of the recursive relations,

$$\mathrm{J}'_{\nu}(u) = +\mathrm{J}_{\nu-1}(u) - \frac{\nu}{u}\mathrm{J}_{\nu}(u) \quad , \tag{7}$$

$$\mathrm{K}'_{\nu}(w) = -\mathrm{K}_{\nu-1}(w) - \frac{\nu}{w}\mathrm{K}_{\nu}(w) \quad , \tag{8}$$

and show that Eq. (6) implies:

$$\frac{u\mathrm{J}_{\nu-1}(u)}{\mathrm{J}_{\nu}(u)} = -\frac{w\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_{v}(w)} \tag{9}$$

Solution:

Inserting Eqs. (7) and (8) into Eq. (6) we get

$$\begin{aligned}
\frac{u\left(\mathrm{J}_{\nu-1}(u) - \frac{\nu}{u}\mathrm{J}_{\nu}(u)\right)}{\mathrm{J}_{\nu}(u)} &= \frac{w\left(-\mathrm{K}_{\nu-1}(w) - \frac{\nu}{w}\mathrm{K}_{\nu}(w)\right)}{\mathrm{K}_{\nu}(w)} \\
\frac{u\mathrm{J}_{\nu-1}(u) - \mathrm{J}_{\nu}(u)}{\mathrm{J}_{\nu}(u)} &= \frac{-w\mathrm{K}_{\nu-1}(w) - \mathrm{K}_{\nu}(w)}{\mathrm{K}_{\nu}(w)} \\
\frac{u\mathrm{J}_{\nu-1}(u)}{\mathrm{J}_{\nu}(u)} - \nu &= \frac{-w\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_{\nu}(w)} - \nu \\
\frac{u\mathrm{J}_{\nu-1}(u)}{\mathrm{J}_{\nu}(u)} &= -\frac{w\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_{v}(w)}
\end{aligned}$$

Note that for $\nu = 0$ Eq. (9) becomes $\frac{u\mathrm{J}_1(u)}{\mathrm{J}_0(u)} = \frac{w\mathrm{K}_1(w)}{\mathrm{K}_0(w)}$. This is because of the symmetry properties of the Bessel function $J_{-\nu}(u) = (-1)^{\nu} J_{\nu}(u)$, and the modified Bessel function $K_{-\nu}(w) = K_{\nu}(w)$.

For each index ν the latter equation can be solved for β, as done already for the slab waveguide. Since the Bessel function oscillates, different solutions are obtained and can be classified by means of a new integer, μ. The normalized cutoff frequencies $V_{\mu,\nu,c}$ of the different modes are obtained from Eq. (9) when we set $w \to 0$ (and simultaneously $u \to V = ak_0\sqrt{n_1^2 - n_2^2}$). From standard properties of the Bessel functions, it can be proven that $\lim_{w\to 0} \frac{w\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_{\nu}(w)} = 0$. The normalized cut-off frequency of the $\mathrm{LP}_{\nu,\mu}$ mode ($\mu = 1, 2, 3..$) is hence determined by the $\mu-$th zero $j_{\nu-1,\mu}$ of the Bessel function $\mathrm{J}_{\nu-1}(u)$.

$$V_{\mu,\nu,\mathrm{c}} = j_{\nu-1,\mu} \tag{10}$$

g) A typical standard single mode fiber has the following specificationss: $a = 4.1\,\mu\mathrm{m}$, $\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} = 0.0035$ and $n_1 = 1.41$. This fiber always supports the fundamental mode $\mathrm{LP}_{0,1}$. The next higher order mode is the $\mathrm{LP}_{1,1}$. What is the minimum wavelength for which the fiber is single-mode? Hint: $j_{0,1} \approx 2.4048$.

Solution:

The normalized cut-off frequency of the $\mathrm{LP}_{1,1}$ mode is given by $V_{1,1,\mathrm{c}} = j_{0,1} = 2.4048$. For the SMF28, this translates into the wavelength according to:

$$\begin{aligned}
V &= a\frac{2\pi}{\lambda_0}\sqrt{n_1^2 - n_2^2} \\
\lambda_c &= a\frac{2\pi}{V_{1,1,c}}\sqrt{n_1^2 - n_2^2} \\
\lambda_c &= 4.1\,\mathrm{\mu m}\frac{2\pi}{2.4048}\sqrt{\Delta \cdot 2n_1^2} = 1.2637\mathrm{\mu m}
\end{aligned}$$

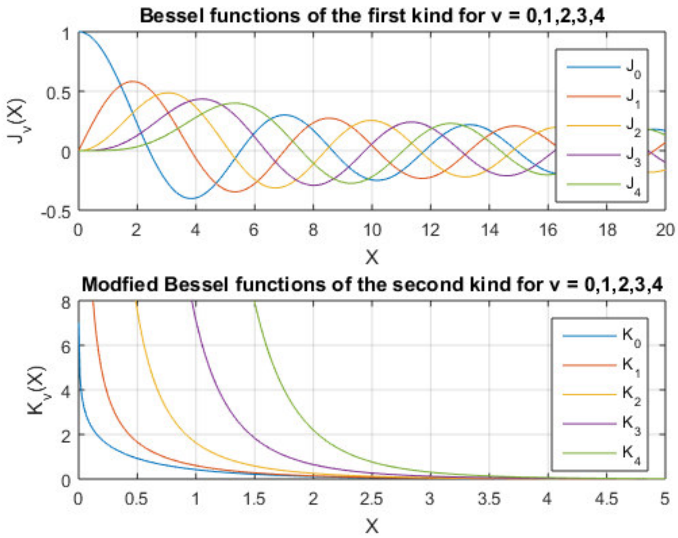

Figure 1: Bessel functions of the first kind and modified Bessel functions of the second kind.

Questions and Comments:

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